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GROUND AND EXCITED STATE CHARACTERISTICS OF THE NUCLEI WITH A = 6

The binding energy, the root-mean-square radius, the magnetic dipole moment, the electric quadrupole moment, and the moment of inertia of the nucleus ^{6}Li are calculated by applying different models. The translation invariant shell model is applied to calculate the binding energy, the root-mean-square radius, and the magnetic dipole moment by using two- and three-body interactions. Also, the spectra of the nuclei with A = 6 are calculated by using the translation-invariant shell model. Moreover, the *ft*-value of the allowed transition: $^6\text{He}\left\{J^{\pi}=0^{+};T=1\right\}\beta^{-}\rightarrow{}^6\text{Li}\left\{J^{\pi'}=1^{+};T'=0\right\}$ is also calculated. Furthermore, the concept of the single-particle Schrödinger fluid for axially symmetric deformed nuclei is applied to calculate the moment of inertia of ^{6}Li. Also, we calculated the magnetic dipole moment and the electric quadrupole moment of the nucleus ^{6}Li in this case of axially symmetric shape. Moreover, the nuclear superfluidity model is applied to calculate the moment of inertia of ^{6}Li, based on a single-particle deformed anisotropic oscillator potential added to it a spin-orbit term and a term proportional to the square of the orbital angular momentum, as usual in this case. The single-particle wave functions obtained in this case are used to calculate the magnetic dipole moment and the electric quadrupole moment of ^{6}Li.

Keywords: translation invariant shell model, nuclei with A = 6, binding energy, spectrum, root-mean-square radius, magnetic dipole moment, quadrupole moment, *ft*-value, single-particle Schrödinger fluid, nuclear superfluidity model.

1. Introduction

Calculations of the ground- and the excited-state characteristics of the nucleus ^{6}Li have been the subject of much research [1 - 16]. This is because the nucleus ^{6}Li has the most distinguished structure among all the *p*-shell nuclei. Two valence nucleons in the *p*-shell are weakly coupled to the core consisting of the four *s*-shell nucleons. The correlation between the two valence nucleons plays the most important role. If, as might be expected from the simple scheme for filling shells, ^{6}Li had the configuration $^4\text{He}\ 1p_{3/2}^2$, according to the empirical rule for forming moments in odd-odd nuclei, its spin should be 3. It may be assumed that this empirical rule is not obeyed; in the ^{6}Li nucleus the spins of the two nucleons add up in such a way as to form a total spin of 1. This system could have a magnetic dipole moment of 0.63 (N.M.) whereas the experimental value is 0.82 N.M. On the other hand, two nucleons in a $1p_{3/2}^2$ the state should have a large quadrupole moment, whereas the experimental value is -0.083 e barns. It is apparent that in the ^{6}Li nucleus we are dealing with an irregularity in the system for filling levels. For these reasons, many models have been applied for this nucleus. Among these models are: the intermediate-coupling shell model [1, 2], the cluster model [3 - 5], the α-deuteron model [6], the shell model and the large space shell model [7 - 9], the no-core shell model (NCSM) [10, 11], the large-basis shell-model (LBSM) [12], and the translation invariant-shell model (TISM) [13 - 16].

The NCSM [10, 11] is an ab initio configuration-interaction (CI) method that has achieved a good description of the low-lying states and nuclear reactions up through *p*-shell nuclei. This method uses the Lanczos algorithm to compute a few lowest-lying eigenstates and eigenvalues of a realistic Hamiltonian matrix whose elements are calculated in an *m*-scheme basis, i.e., basis of the Slater determinants constructed from single-particle wave functions of the harmonic oscillator. The NCSM and the LBSM are very close to the TISM and the three models are applied successfully to the ground and the excited state characteristics of the nucleus ^{6}Li.

The TISM considers the nucleus as a system of non-interacting quasi-particles and enables us to apply the algebraic methods for calculating the matrix elements of operators that correspond to physical quantities. The bases of this model are constructed in such a way that they will have a certain symmetry with respect to the interchange of particles and have definite total angular momentum. This model has shown good results for the structure of light nuclei [17 - 23].

The large quadrupole moments observed in some nuclei, which do not belong to closed shells, implied a collective deformation and thereby a rotational degree of freedom [24, 25]. The most central parameter of collective rotation is the moment of inertia of deformed nuclei [25, 26]. The study of the velocity fields for the rotational motion of the axially symmetric deformed nuclei led to the formulation of the so-called Schrödinger fluid [27, 28] and to a simple method for calculating the moments of inertia of axially deformed nuclei [29, 30].

In a previous paper [31], Doma presented the results of calculating the binding energy (B. E.), the spectrum, the root-mean-square radius, the magnetic dipole moment, the electric quadrupole moment, and the moment of inertia of the nucleus ^{6}Li by using truncated bases of the USM corresponding to $N \le 8$ and using two-body potentials only.

In the present paper, the ground- and the excited-state wave functions of the nuclei with A = 6 are expanded in series in terms of the TISM basis functions corresponding to the number of quanta of excitation $2 \le N \le 11$ in order to calculate the ground and the excited state characteristics of these nuclei. For the nucleon-nucleon interaction, we used the Gogny, Pires, and De Tourreil (GPT) interaction [32] and the Av8$'$ potential [33, 34]. The GPT potential is a smooth realistic local nucleon-nucleon potential suitable for Hartree - Fock (HF) calculations. It gives an acceptable fit to two nucleon data up to 300 MeV and reasonable properties for finite nuclei, particularly the radii, for the HF approximation. It consists of central, spin-orbit, tensor, and quadratic spin-orbit terms. The radial dependences are represented by sums of Gaussian functions. The potential of this type is particularly suited for calculations of nuclear states and reactions because it can be handled easily. Especially, it simplifies calculations in the harmonic oscillator shell model. The Av8$'$ potential is derived from the realistic AV18 interaction by neglecting the charge dependence and the terms proportional to L^2 and (L.S)^2. Furthermore, in this paper, we omitted the electromagnetic part of the interaction. This potential is local, and its spin and isospin dependencies are represented by operators. For the three-nucleon interaction, we used the Urbana UIX-potential [34, 35]. Accordingly, the energy eigenvalues and the corresponding eigenfunctions of the different states of the nuclei with A = 6 are calculated as functions of the oscillator parameter $\hbar\omega$, which is varied in a wide energy range in order to obtain the best spectrum of the nucleus. The ground-state nuclear wave functions of ^{6}Li which are obtained from the diagonalization of the ground-state Hamiltonian matrices are used to calculate the root-mean-square radius of ^{6}Li as a function of $\hbar\omega$ by using the two nucleon-nucleon interactions alone, and by adding to them the three-nucleon interaction. Furthermore, the nuclear supermultiplet model [36] is then applied to calculate the magnetic dipole moment of ^{6}Li by using the bases of the TISM. Moreover, the *ft*-value of the allowed transition: ^{6}He $\left\{J^{\pi}=0^{+}; T=1\right\}\beta^{-} \to {}^6\mathrm{Li}\left\{J^{\pi'}=1^{+}; T'=0\right\}$ is also calculated.

Also, we applied the concept of the single-particle Schrödinger fluid to construct the ground state of the nucleus ^{6}Li by assuming that this nucleus is deformed and has an axis of symmetry. Accordingly, the cranking-model moment of inertia and the rigid-body moment of inertia of the nucleus ^{6}Li are calculated as functions of the deformation parameter β and the non-deformed oscillator parameter $\hbar\omega_0^0$, which is given in terms of the mass number A, the number of protons Z and the number of neutrons N [37]. We also calculated the magnetic dipole moment and the electric quadrupole moment of the nucleus ^{6}Li in this case of axially symmetric shape.

Furthermore, we finally considered a single-particle deformed potential consisting of an anisotropic oscillator potential added to it a spin-orbit term and a term proportional to the square of the orbital angular momentum of the nucleon to calculate the single-particle energy eigenvalues and eigenfunctions for a nucleon in a deformed non-axial nucleus [38]. As a consequence, the ground-state of the nucleus ^{6}Li is constructed and its moment of inertia is calculated by applying the superfluidity nuclear model of Belyaev [39] as a function of the deformation parameter β, the non-axiality parameter γ, and the non-deformed oscillator parameter $\hbar\omega_0^0$. The single-particle wave functions which are obtained in this case are used to calculate the magnetic dipole moment and the electric quadrupole moment of the nucleus ^{6}Li.

2. The spectrum, the root-mean square radius, and the magnetic dipole moment

In the TISM, the Hamiltonian of a nucleus with mass number A, and two- and three-body interactions are usually written as [22, 23]

$$H = H^{(0)} + V' + V'', \tag{2.1}$$

where

$$H^{(0)} = \frac{1}{\mathrm{A}} \sum_{1=i<j}^{\mathrm{A}} \left[\frac{(\boldsymbol{p}_i - \boldsymbol{p}_j)^2}{2m} + \frac{1}{2} m\omega^2 (\boldsymbol{r}_i - \boldsymbol{r}_j)^2 \right] \tag{2.2}$$

is the TISM-Hamiltonian,

$$V' = \sum_{1=i<j}^{\mathrm{A}} \left[V\left(\left|\boldsymbol{r}_i - \boldsymbol{r}_j\right|\right) - \frac{m\omega^2}{2} \left(\boldsymbol{r}_i - \boldsymbol{r}_j\right)^2 \right] = \sum_{1=i<j}^{\mathrm{A}} V'_{ij} \tag{2.3}$$

is the residual nucleon-nucleon interaction, and

$$V'' = \sum_{1=i<j<k}^{A} V''_{ijk} \qquad (2.4)$$

is the three-body interaction. The eigenfunctions and eigenvalues of the Hamiltonian $H^{(0)}$ are given by [14 - 16, 22, 23]

$$\varphi_{\alpha_1 i_1,\ldots,\alpha_N i_N} = C\, a^{+}_{\alpha_1 i_1} a^{+}_{\alpha_2 i_2} \ldots a^{+}_{\alpha_N i_N} \exp\left\{-\frac{m\omega}{2\hbar}\sum_{\varepsilon=1}^{3}\sum_{i=1}^{A-1}\xi^{2}_{\varepsilon i}\right\} \qquad (2.5)$$

and

$$E^{(0)}_{N} = \left[N + \frac{3}{2}(A\text{-}1)\right]\hbar\omega. \qquad (2.6)$$

Because of the symmetry properties of the functions (2.5), they can be used as bases for irreducible representations (IRs) of symmetric tensors of the rank N. The functions (2.5) are symbolically denoted by [14, 22, 23]

$$\left|A\,\Gamma_L M_L;\ \Gamma_S M_S M_T\right\rangle \equiv \left|A\,N\{\rho\}(\upsilon)\alpha[f](\lambda\mu)LM_L;\ [\tilde{f}]STM_S M_T\right\rangle, \qquad (2.7)$$

where Γ_L and Γ_S are the sets of all orbital and spin-isospin quantum numbers, respectively. In terms of these functions, one can construct bases with total angular momentum quantum number J, total isotopic spin quantum number T, and parity π as follows [14]

$$\left|AJ^{\pi}TM_J M_T\right\rangle = \sum_{\Gamma} C^{J^{\pi}T}_{\Gamma} \sum_{M_L+M_S=M_J} \left(LM_L, SM_S|JM_J\right)\left|A\Gamma_L M_L; \Gamma_S M_S M_T\right\rangle, \qquad (2.8)$$

where Γ is the set of all quantum numbers in Γ_L and Γ_S and $(LM_L, SM_S|JM_J)$ are Clebsch - Gordan coefficients of the rotational group SO_3. The coefficients $C^{J^{\pi}T}_{\Gamma}$ in Eq. (2.8) are the state-expansion coefficients, where the number of quanta of excitations N is allowed to be either even or odd integer depending on the parity of the state π. It is seen from Eq. (2.2) that the Hamiltonian $H^{(0)}$ is free of spurious states, which correspond to the nonzero motion of the center of mass of the whole nucleus.

Concerning the two-body potential, given by $V(|\mathbf{r}_i - \mathbf{r}_j|)$, we used the GPT-potential [32] and the Av8'-potential [33, 34]. The two potentials when used for the nuclei with mass numbers $A = 3$, 4, and 7 gave good results for their ground and excited state properties [16, 17, 20 - 23]. For the three-body potential, we used the Urbana UIX potential [34, 35]. The methods of calculating the one-, the two- and the three-particle fractional parentage coefficients (FPCs) are given in [36, 40, 41]. Also, the methods of calculating the matrix elements of the nuclear characteristics by using the TISM are given in [14, 15, 20 - 23]. Accordingly, we calculated the energy eigenvalues and eigenfunctions of the ground- and the excited states of ^{6}Li.

The root-mean-square radius is calculated from the relation [14, 17, 36]

$$R = \sqrt{r_p^2 + \left(R^2_{Nuc}\right)}. \qquad (2.9)$$

In Eq. (2.9), $r_p = 0.85$ fm is the proton radius and the second term under the root is the expectation value of the operator

$$R^2_{Nuc} = \frac{1}{A^2}\sum_{1=i<j}^{A} r^2_{ij}. \qquad (2.10)$$

The detailed methods of calculating R can be found in [14, 15] by using the bases of the TISM.

The nuclear magnetic dipole moment μ is defined as the expectation value of the operator $\hat{\mu}$, which can be written in the form [14, 36]

$$\hat{\mu} = \hat{\mu}_o + \hat{\mu}_\sigma, \qquad (2.11)$$

where $\hat{\mu}_o$ is the orbital part, given by

$$\hat{\mu}_o = \frac{1}{2}\sum_{i=1}^{A}(1 - 2t_{oi})\ell_{oi}, \qquad (2.12)$$

and the spin-isospin part $\hat{\mu}_o$ is given by

$$\hat{\mu}_\sigma = \sum_{i=1}^{A}\left[(\mu_p + \mu_n) + 2(\mu_p - \mu_n)t_{oi}\right]s_{oi}, \qquad (2.13)$$

calculated in a state with $M_J = J$. In equations (2.12) and Eq. (2.13), ℓ_{0i}, s_{0i} and t_{0i} are the z-components of the orbital angular momentum, the spin, and the isotopic spin of the i^{th}-nucleon, respectively. μ_p and μ_n are the proton and the neutron magnetic moments, respectively. The method of calculating the nuclear magnetic dipole moment can be found in [23].

3. The *ft*-value of the allowed β⁻-transition

The allowed β⁻-decay is characterized by the quantity *ft*, which is related to the transition by the following simple relation [14, 21, 36]

$$ft = \frac{6200}{\mathfrak{S}^2 + 1.41\mathfrak{M}^2}\, s. \qquad (3.1)$$

The quantities $\mathfrak{S}^2$ and $\mathfrak{M}^2$ are given in terms of the matrix elements of the two operators

$$\mathfrak{S} = \mp\sqrt{2}\sum_{i=1}^{A} t_{\pm1}^{1}(i), \qquad (3.2)$$

and

$$\mathfrak{M}_{\pm1\mu}^{11} = \mp 2\sqrt{2}\sum_{i=1}^{A} t_{\pm1}^{1}(i)\, s_{\mu}^{1}(i), \qquad (3.3)$$

respectively. These operators are invariant with respect to the group SU_2. The operators which define the allowed β⁻-decay, and hence their matrix elements, do not depend on the orbital coordinates of the wave function, i.e., on most of the quantum numbers of this nuclear wave function. For this reason, it is appropriate to use the approximations of the nuclear supermultiplet model [20 - 22, 36].

The matrix elements of the operator given by Eq. (3.2) depend only on the total isotopic spin T and its z-component M_T of the ground state wave function and it is easy to prove that

$$\mathfrak{S}^2 \equiv \left| TM_T |\mathfrak{S}| T'M_T' \right|^2 = \left[T(T+1) - M_T M_T' \right] \delta_{T'M_{T\pm1}'}^{TM_T}. \qquad (3.4)$$

It is noted that the matrix elements in Eq. (3.4) are diagonal with respect to all the other quantum numbers.

The operator $\mathfrak{M}_{\pm1\mu}^{11}$ represents the spin and the isospin coordinates of the nucleons, and can be written in an irreducible form relative to the conversions of the symmetric group S_A in the following form

$$\mathfrak{M}_{\pm1\mu}^{11} = \mp 2\sqrt{2}\sum_{\chi q} \mathfrak{S}_{\pm1q}^{1\chi}\, \mathfrak{H}_{\mu q}^{1\chi}, \qquad (3.5)$$

where $\mathfrak{S}_{\pm1q}^{1\chi}$ and $\mathfrak{H}_{\mu q}^{1\chi}$ are irreducible tensors of the groups SU_2 and S_A, respectively, χ takes the IRs $[\mathrm{A}]$ and $[\mathrm{A}-1,1]$, and q is its projection. To calculate the matrix elements of the operator (3.5), we apply the Wigner - Eckart theorem for the two groups simultaneously.

In the present calculation of the *ft*-value, we are concerned with the following transition

$${}^{6}\mathrm{He}\left\{J^{\pi} = 0^{+}; T = 1\right\}\beta^{-} \to {}^{6}\mathrm{Li}\left\{J^{\pi'} = 1^{+}; T' = 0\right\}.$$

The method of calculating the required matrix elements of the quantity *ft* can be found in [14, 21].

4. The nuclear moment of inertia and the quadrupole moment

According to the concept of the single-particle Schrödinger fluid for axially symmetric deformed nuclei, the cranking model, and the rigid-body model moments of inertia of a given nucleus are calculated by the following expressions [28 - 30]

$$\mathfrak{I}_{cr} = \frac{E}{\omega_o^2}\frac{1}{6+2\sigma}\left(\frac{1+\sigma}{1-\sigma}\right)^{\frac{1}{3}}\left[\sigma^2(1+q) + \frac{1}{\sigma}(1-q)\right], \qquad (4.1)$$

$$\mathfrak{I}_{rig} = \frac{E}{\omega_o^2}\frac{1}{6+2\sigma}\left(\frac{1+\sigma}{1-\sigma}\right)^{\frac{1}{3}}\left[(1+q) + \sigma(1-q)\right], \qquad (4.2)$$

where q is the anisotropy of the configuration, which is defined by

$$q = \frac{\sum_{occ}\left(n_y + \frac{1}{2}\right)}{\sum_{occ}\left(n_z + \frac{1}{2}\right)}, \qquad (4.3)$$

and E is the total energy of the oscillator

$$E = \sum_{occ}\left[\hbar\omega_y\left(n_y + n_x + 1\right) + \hbar\omega_z\left(n_z + \frac{1}{2}\right)\right]. \qquad (4.4)$$

In Eqs. (4.3) and (4.4) n_x, n_y and n_z are the state quantum numbers of the oscillator. Also, in Eqs. (4.1) and (4.2) σ is a measure of the deformation of the potential and is defined by

$$\sigma = \frac{\omega_y - \omega_z}{\omega_y + \omega_z}. \qquad (4.5)$$

In the above equations, we use the well-known Nilsson angular frequencies [42]

$$\omega_x^2 = \omega_y^2 = \omega_0^2\left(1+\frac{2\delta}{3}\right), \qquad (4.6)$$

$$\omega_z^2 = \omega_0^2\left(1-\frac{4\delta}{3}\right), \qquad (4.7)$$

where δ is a deformation parameter related to the well-known deformation parameter β by

$$\delta = \frac{3}{2}\sqrt{\frac{5}{4\pi}}\beta. \qquad (4.8)$$

The frequency ω_0 in Eqs. (4.6) and (4.7) is given in terms of the non-deformed frequency ω_0^0 by [42]

$$\omega_0 = \omega_0(\delta) = \omega_0^0\left(1-\frac{4}{3}\delta^2-\frac{16}{27}\delta^3\right)^{-\frac{1}{6}}. \qquad (4.9)$$

The value of the non-deformed oscillator parameter $\hbar\omega_0^0$ depends on the mass number A, the number of neutrons N, and the number of protons Z. An approximate formula for $\hbar\omega_0^0$ is given by [37]

$$\hbar\omega_0^0 = \frac{38.6\mathrm{A}^{-\frac{1}{3}}}{\left[1+\frac{1.646}{\mathrm{A}}-\frac{0.191(\mathrm{N}-Z)}{\mathrm{A}}\right]^2}. \qquad (4.10)$$

In this case of axially symmetric deformed nuclei, the magnetic dipole moment can be calculated using the same technique given by Nilsson [42].

Furthermore, assuming a charge distribution in accordance with the Thomas - Fermi statistical model, one obtains, for the case of axially symmetric deformed nuclei, the intrinsic quadrupole moment [42]

$$Q_0 = 0.8\, ZeR^2\delta\left(1+\frac{2\delta}{3}\right), \qquad (4.11)$$

where Z is the number of protons and R is the radius of charge of the nucleus. In (4.11) Q_0 is calculated to the second order in the deformation parameter δ. The relation between the measured quadrupole moment, denoted by Q_S, and Q_0 is given by

$$Q_S = \frac{3K^2 - I(I+1)}{(I+1)(2I+3)}Q_0, \qquad (4.12)$$

where I is the total spin of the nuclear state and K is its component along the body-fixed z-axis. The intrinsic quadrupole moment is then calculated for a nucleus with an axis of symmetry by calculating the charge radius using the single-particle wave functions, as a function of the deformation parameter δ, and hence the measured quadrupole moment is obtained.

5. The nuclear moment of inertia and the quadrupole moment when the nucleus does not have an axis of symmetry

Consider a nucleon that is moving in a deformed nuclear field whose Hamiltonian is given by [38]

$$H = -\frac{\hbar^2}{2m}\nabla^2 + \frac{m}{2}\omega_0^2 r^2 + C\boldsymbol{l}.\boldsymbol{s} + Dl^2 -$$
$$-m\omega_0^2 r^2\beta cos\gamma Y_{2,0}(\theta,\varphi) -$$
$$-\frac{\sqrt{2}}{2}m\omega_0^2 r^2\beta sin\gamma\left\{Y_{2,2}(\theta,\varphi)+Y_{2,-2}(\theta,\varphi)\right\}, \quad (5.1)$$

where $Y_{l,\Lambda}(\theta,\varphi)$ are the spherical harmonics, β is the deformation parameter and γ is the non-axiality parameter. The constants C and D in equation Eq. (5.1) are given by [38, 42]

$$C = -2\chi\hbar\omega_0^0, \quad D = -\mu\chi\hbar\omega_0^0, \qquad (5.2)$$

where χ takes values in the interval $0.05 \le \chi \le 0.08$ and μ depends on the number of quanta of excitation N as given by Nilsson [42]. The Schrödinger equation representing the motion of a single nucleon in the non-axially deformed nuclear field, whose Hamiltonian operator is given by Eq. (5.1), can be solved [38] by: (i) applying the variational method for the fifth term in Eq. (5.1) with respect to the eigenfunctions of the first four terms $|Nl\Lambda\Sigma\rangle$, and then (ii) applying the stationary non-degenerate perturbation method for the last term in (5.1) with respect to the eigenfunctions $|N\Omega^\pi\rangle$ which results from the application of the variational method. As a result, the single-particle energy eigenvalues and eigenfunctions $|\Omega^\pi\rangle$ of a nucleon in a deformed nuclear field can be calculated for every level, with the given value of the z-component of the total angular momentum Ω and parity π as functions of the potential parameters χ, and μ, the deformation parameter β, and the non-axiality parameter γ.

Hence, the moment of inertia of a deformed nucleus that does not have an axis of symmetry is then given by applying the superfluidity model [38, 39]

$$\Im_{s.f.} = \hbar^2\sum_{i,k}\frac{\langle i|J_x|k^2\rangle}{E_i+E_k}\left\{1-\frac{(\zeta_i-\lambda)(\zeta_k-\lambda)+\Delta^2}{E_iE_k}\right\}, \qquad (5.3)$$

where ζ_i are the eigenvalues of the self-consistent field, the eigenvalues of the Hamiltonian operator (5.1), λ is the chemical potential and the energy of elementary excitations of the nucleus, E_i, is given by

$$E_i = \sqrt{(\zeta_i - \lambda)^2 + \Delta^2}. \qquad (5.4)$$

In Eq. (5.4), Δ is the energy gap and λ is the chemical potential given by [38]

$$\sum_i \left\{ 1 - \frac{\zeta_i - \lambda}{\sqrt{(\zeta_i - \lambda)^2 + \Delta^2}} \right\} = \mathrm{N}_{p,n}, \qquad (5.5)$$

where the summation, here, runs overall distinct neutron (or proton) energies and $\mathrm{N}_{p,n}$ is the number of protons or neutrons inside the nucleus.

In this case of non-axial deformed nuclei, the intrinsic quadrupole moment of a nucleus, consisting of Z protons, is given by

$$Q_0 = \sum_{i=1}^{Z} Q_i, \qquad (5.6)$$

where the single-particle operator Q_i is given by

$$Q_i = e\sqrt{\frac{16\pi}{5}} \int \left(\Psi^i_{\Omega\pi}\right)^2 r_i^2 Y_{2,0}(\theta_i, \phi_i)\, d\tau. \qquad (5.7)$$

Carrying out the integration in Eq. (5.7) with respect to the wave functions $|\Omega^\pi\rangle$, one then obtains

$$Q_i = e\sqrt{\frac{16\pi}{5}} \sum_{k,m} C_k^i C_m^i \left\langle N_k l_k \left| r^2 \right| N_m l_m l_k \Lambda_k \left| Y_{2,0} \right| l_m \Lambda_m \right\rangle. \qquad (5.8)$$

Filling the single-particle wave functions $|\Omega^\pi\rangle$ for the nucleus ^{6}Li in its ground-state with two protons and two neutrons in the 0s-level and one proton and one neutron in the 1p-level, it is then possible to calculate the quadrupole moment by calculating the necessary matrix elements of Eq. (5.8) and evaluating the expansion coefficients C_k^i of the functions $|\Omega^\pi\rangle$ in terms of the functions $|Nl\Lambda\Sigma\rangle$.

6. Results and discussions

In the present paper, we applied the TISM with a large number of bases belonging to the number of quanta of excitation $2 \le N \le 11$ to calculate the B. E., the root-mean-square radius, and the magnetic dipole moment of the nucleus ^{6}Li. Furthermore, the spectrum of the nuclei with A = 6 and the *ft*-value of the allowed transition: ^{6}He $\{J^\pi = 0^+; T = 1\}\beta^- \to \to$ ^{6}Li $\{J^{\pi'} = 1^+; T' = 0\}$ are also calculated. In the calculations, which have been carried out in this part, we used two nucleon-nucleon interactions, namely: the GPT [32] and the Av8' [33, 34] together with the Urbana IX three-body interaction [34, 35]. To our knowledge, no one reached this large basis TISM calculations for the nuclei with A = 6 before.

In Fig. 1 we present the variation of the B. E. of ^{6}Li, in MeV, with respect to the oscillator parameter $\hbar\omega$ by using the GPT and the Av8' two-body potentials and the improved values obtained by adding the Urbana UIX three-body potential. From this figure, we notice that all the potentials produced maximum values (minimum ground-state energy) as should be expected from the behavior of the TISM. Also, the results obtained by using the Av8' potential together with the UIX potential are better than those obtained by using the other potentials. Moreover, the variation of the root-mean-square radius (R) of ^{6}Li with respect to the oscillator parameter $\hbar\omega$ is given in Fig. 2, for the three potentials. It is seen from this figure also that all the potentials have minimum values and that the Av8' potential together with the UIX potential gave the best values of R among the other potentials.

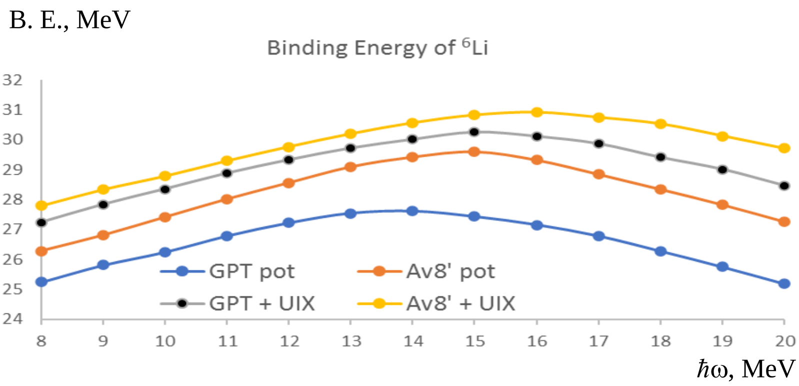

Fig. 1. Variation of the B. E. of ^{6}Li with $\hbar\omega$ for the used potentials. (See color Figure on the journal website.)

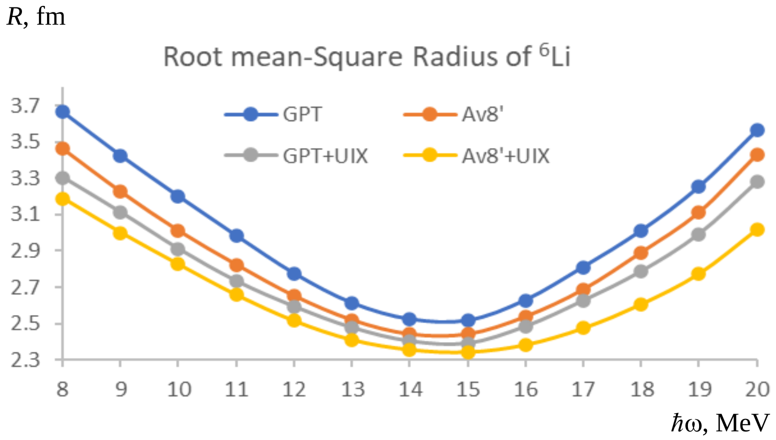

Fig. 2. Variation of the root-mean-square radius of ^{6}Li with $\hbar\omega$ for the used potentials.
(See color Figure on the journal website.)

In Table 1, we present the values of the B. E., the root-mean-square radius (R), the magnetic dipole moment (μ) of the nucleus ^{6}Li, and the *ft*-value of the allowed transition: $^6\text{He}\ \left\{J^{\pi}=0^{+};T=1\right\}\beta^{-}\rightarrow$ $\rightarrow\ ^6\text{Li}\ \left\{J^{\pi'}=1^{+};T'=0\right\}$ by using the three potentials. The corresponding experimental values are given in this table. Also, in Table 1 we present the values of $\hbar\omega$, for which the spectra of the nuclei with A = 6 are in good agreement with the corresponding experimental values. Previous results obtained by using the Doma-potential (D1 [16]) with basis-functions of the TISM belonging to $N \leq 6$ [16] are also given.

Table 1. **B. E., root-mean-square radius, and magnetic dipole moment of ^{6}Li and the *ft*-value of the allowed β⁻-decay**

Case	B. E., MeV	*R*, fm	μ, N.M.	*ft*-value, s	ħω, MeV
GPT	27.634	2.491	0.836	886.94	14
Av8'	29.744	2.361	0.839	885.17	15
GPT + UIX	30.323	2.392	0.828	881.19	15
Av8' + UIX	30.914	2.351	0.832	880.22	16
TISM + D1 [16]	24.572	2.147	0.782	–	20
Exp. [36]	32.0	2.32	0.822	862 ± 17	–

The spectra of the even and the odd parity states of the nuclei with A = 6 are given in Table 2, by using the three potentials. The corresponding experimental values [44] and previous values obtained by using the NCSM with the CD-Bonn potential [45] are also given in Table 2.

Table 2. **The spectra of the even- and the odd-parity states of the nuclei with A = 6. The energies are in MeV**

J^{π},T	Exp. [44]	GPT	Av8'	GPT + UIX	Av8' + UIX	NCSM CD-Bonn [45]
$3^{+},0$	2.186	2.231	2.444	2.204	2.411	2.841
$0^{+},1$	3.563	3.397	3.101	3.552	3.397	3.330
$2^{+},0$	4.310	4.373	4.566	4.355	4.309	4.610
$2^{+}_{1},1$	5.366	5.221	5.494	5.288	5.398	5.975
$1^{+}_{2},0$	5.652	6.421	6.533	6.192	5.838	6.544
$2^{+}_{2},1$	6.633	6.883	7.707	6.771	6.999	9.199
$2^{-},0$	6.941	7.222	8.821	7.148	7.418	–
$1^{-},0$	8.732	8.177	9.542	8.456	8.959	–
$0^{-},0$	9.302	9.166	10.431	9.222	9.676	–
$1^{+},1$	–	9.452	10.995	9.303	9.872	9.937

From Table 1 we see that the resulting values of the B. E. and the root-mean-square radius of ^{6}Li are in good agreement with the corresponding experimental values, especially by adding the three-body interaction. Also, the inclusion of states corresponding to the number of quanta of excitations $6 < N \leq 11$ mainly improved the previous results obtained by using the Doma-potential (D1 [16]) with basis-functions of the TISM belonging to $N \leq 6$ [16].

Furthermore, we notice from Table 2 that the obtained spectra of the nuclei with A = 6, for the even- and the odd-parity states, are in good agreement with the corresponding experimental one, especially in the case of the GPT + UIX potentials.

The calculated values of the magnetic dipole moment of the nucleus ^{6}Li, in N.M., by applying the three nuclear models, corresponding to the spherical, the axially symmetric, and the nonaxial deformed cases are given in Table 3.

Table 3. **The magnetic dipole moment of ^{6}Li**

Case	γ	β	$\hbar\omega_0^0$, MeV	μ, N.M.
Spherical (GPT + UIX)	0^0	0.0	14	0.828
Spherical (Av8'+UIX)	0^0	0.0	15	0.832
Axially Symmetric	0^0	0.26	9.594	0.826
Asymmetric	30^0	0.28	9.594	0.939
Experimental	–	0.20 - 0.26	–	0.822 [36]

The calculated values of the electric quadrupole moment of ^{6}Li, in em barns, are given in Table 4 for the axially symmetric case ($\gamma = 0^0$) and for the non-axial case corresponding to $\gamma = 30^0$.

Table 4. **The electric quadrupole moment of ^{6}Li**

Case	γ	β	$\hbar\omega_0^0$, MeV	Q_S, e m barns	Q_{exp}, e m barns [46]
Symmetric	0^0	–0.06	9.594	–0.078	–0.083
Asymmetric	30^0	–0.12	9.594	–0.074	–0.083

In Table 5, we present the calculated values of the reciprocal moment of inertia of the nucleus ^{6}Li by using the concept of the single-particle Schrödinger fluid, for the axially symmetric case, for both cranking and the rigid-body models, and the nuclear superfluidity model for the non-axial case. Also, we present in Table 5 the corresponding experimental value. The values of the deformation parameter β, the non-axiality parameter γ, and the oscillator parameter $\hbar\omega_0^0$ are also given in Table 5.

Table 5. **Reciprocal moment of inertia of ^{6}Li**

Case	γ	β	$\hbar\omega_0^0$, MeV	$\frac{\hbar^2}{2\Im}$, keV
Cranking	0^0	0.27	9.594	493.44
Rigid body	0^0	0.24	9.594	716.04
Superfluidity	25^0	0.28	9.594	548.58
Experimental	–	0.20 - 0.26	–	500.0 [47]

In the case of the superfluidity model, the best values of the model parameters are: $\chi = 0.08$, $\lambda = 18.121$ MeV, and $\Delta = 0.833$ MeV, for the potential parameter, the chemical potential, and the energy gap, respectively. These parameters gave a reciprocal moment of inertia of ^{6}Li in better agreement with the corresponding experimental value.

It is seen from Table 5 that the calculated value of the cranking-model reciprocal moment of inertia is in better agreement with the corresponding experimental value rather than the other values. The disagreement between the value of the rigid-body reciprocal moment of inertia and the corresponding experimental value is because the pairing correlation is not taken into concern in this model [26, 30]. Moreover, it is seen from Table 4 that the calculated value of the quadrupole moment of ^{6}Li in the case $\gamma = 0^0$, the axially symmetric case, is in better agreement with the corresponding experimental value rather than that of the case $\gamma = 30^0$.

7. Conclusions

In the first part of this paper, we considered the nuclei with mass number A = 6 as spherical nuclei and applied the TISM with the number of quanta of excitation $2 \le N \le 11$. In the calculations which have been carried out in this case, we used two nucleon-nucleon potentials together with a three-nucleon potential. Accordingly, we calculated the B. E., the root-mean-square radius, and the magnetic dipole moment. Also, the spectra of the nuclei with A = 6 are calculated. Moreover, the *ft*-value of the allowed transition: $^6\mathrm{He}\left\{J^\pi = 0^+; T = 1\right\}\beta^- \to$ $\to {}^6\mathrm{Li}\left\{J^{\pi'} = 1^+; T' = 0\right\}$ is also calculated. In the second part of this paper, we applied the concept of the single-particle Schrödinger fluid for axially symmetric deformed nuclei to calculate the moment of inertia of ^{6}Li. Also, we calculated the magnetic dipole moment and the electric quadrupole moment of the nucleus ^{6}Li in this case of axially symmetric shape. Finally, in the third part of our investigation, we considered the nucleus ^{6}Li as deformed and does not have an axis of symmetry. Accordingly, we applied an anisotropic single-particle oscillator to represent the average potential field of the nucleons inside the nucleus and applied the variational method followed by the stationary nondegenerate perturbation method to calculate the single-particle energy eigenvalues and the eigenfunctions of the nucleon inside this nucleus. Accordingly, we calculated the magnetic dipole moment and the electric quadrupole moment of ^{6}Li in this case. Also, in this case, we applied the nuclear superfluidity model to calculate the moment of inertia. This study has identified that all the applied models and methods are appropriate and correctly describe the nuclei with mass number A = 6. The second major finding was that the inclusion of states corresponding to the number of quanta of excitations $6 < N \le 11$ mainly improved the previous results obtained by using the Doma-potential (D1 [16]) with basis-functions of the TISM belonging to $N \le 6$ [16]. Furthermore, the inclusion of the UIX- three-nucleon interaction mainly improves the results of the ground-state characteristics of the nucleus ^{6}Li, as well as the spectra of the nuclei with mass number A = 6. Moreover, in the case where the nucleus is assumed to be deformed and has an axis of symmetry, the obtained results concerning the moment of inertia, the magnetic dipole moment, and the electric quadrupole moment, are in better agreement with the corresponding experimental values.

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ХАРАКТЕРИСТИКИ ОСНОВНОГО ТА ЗБУДЖЕНИХ СТАНІВ ЯДЕР З A = 6

Енергія зв'язку, середньоквадратичний радіус, магнітний дипольний момент, електричний квадрупольний момент та момент інерції ядра ^{6}Li були обчислені за допомогою різних моделей. Трансляційно-інваріантна оболонкова модель була застосована для обчислення енергії зв'язку, середньоквадратичного радіуса та магнітного дипольного моменту з використанням дво- та тричастинкових взаємодій. Також спектри ядер з A = 6 були обчислені в трансляційно-інваріантній оболонковій моделі. До того ж було розраховане значення *ft* для дозволеного переходу $^6\mathrm{He}\left\{J^{\pi}=0^{+};T=1\right\}\beta^{-}\rightarrow{}^6\mathrm{Li}\left\{J^{\pi'}=1^{+};T'=0\right\}$. Для розрахунку моменту інерції ^{6}Li була застосована концепція одночастинкової рідини Шредінгера для аксіально-симетричних деформованих ядер. Також було розраховано магнітний дипольний момент та електричний квадрупольний момент ядра ^{6}Li для цього випадку аксіально-симетричної форми. Крім того, модель ядерної надплинності була застосована для обчислення моменту інерції ^{6}Li, базуючись на одночастинковому деформованому анізотропному осциляторному потенціалі з доданим спін-орбітальним членом та членом, пропорційним квадрату орбітального моменту імпульсу, як зазвичай у цьому випадку. Отримані одночастинкові хвильові функції були використані для обчислення магнітного дипольного моменту та електричного квадрупольного моменту ^{6}Li.

Ключові слова: трансляційно-інваріантна оболонкова модель, ядра з A = 6, енергія зв'язку, спектр, середньоквадратичний радіус, магнітний дипольний момент, квадрупольний момент, *ft*-значення, одночастинкова рідина Шредінгера, модель ядерної надплинності.

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